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ARBITRARY TORSION CLASSES AND ALMOST FREE ABELIAN GROUPS[†]

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ABSTRACT

Using the set theoretical principle ∇_κ for arbitrary large cardinals κ, arbitrary large strongly κ-free abelian groups A are constructed such that $\mathrm{Hom}(A, G) = \{0\}$ for all cotorsion-free groups G with $|G| < \kappa$. This result will be applied to the theory of arbitrary torsion classes for Mod-**Z**. It allows one, in particular, to prove that the class **F** of cotorsion-free abelian groups is not cogenerated by a *set* of abelian groups. This answers a conjecture of Göbel and Wald positively. Furthermore, arbitrary many torsion classes for Mod-**Z** can be constructed which are not generated or not cogenerated by single abelian groups.

Introduction

A. Some Notations

Throughout this paper R denotes a ring with identity. All modules are right R-modules. The category of right R-modules is denoted by Mod-R and we write M_R to indicate that M is in this category. $E(M_R)$ is the injective envelope of M_R. $(\mathcal{T}, \subset)$ is the lattice of arbitrary and $(\mathcal{T}_H, \subset)$ the lattice of hereditary torsion classes for Mod-R.

B. Motivation

It is a well known and fundamental fact that a hereditary torsion class **T** for Mod-R is uniquely determined by the family of right ideals I for which R/I is a torsion module or alternatively by the family of right ideals K for which $E(R/K)$ is a torsion free module (cf. [16], chapter vi). This fact has three consequences which were first pointed out by Jans [10] (cf. also Lambek [13], page 6):

(1) **T** is *generated* by a *single* right R-module, namely by the direct sum of all non-isomorphic cyclic torsion modules.

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(2) **T** is *cogenerated* by a *single* right R-module, namely by the direct product of the injective envelopes of the non-isomorphic cyclic torsion free modules.

(3) There exists a cardinal number λ ($\lambda := 2^{2^{|R|}}$) such that the cardinality of $\mathcal{T}_H$ is not greater than λ. We shall say that $\mathcal{T}_H$ has *small cardinality*.

It is very natural to try to extend these results to arbitrary torsion classes for Mod-R. Hence the solution of the following problems is of fundamental interest for the theory of arbitrary torsion classes for Mod-R:

(1) Is *every* torsion class for Mod-R *generated* and *cogenerated* by a single right R-module?

(2) Has $\mathcal{T}$ *small cardinality*?

If we assume a set theory which satisfies for arbitrary large regular cardinals κ the principle ∇_κ which was used by Shelah [15] and Dugas and Göbel [3] to construct strongly κ-free modules (cf. §2 for details) the main result of this paper (Theorem 2.1 and Corollary 2.1) answers — for the ring $\mathbf{Z}$ of integers — both questions *negatively*. Hence $\mathcal{T}$ is generally incalculable. Our Theorem 2.1 also shows that a conjecture of R. Göbel and B. Wald [7] is true: The class of cotorsion-free abelian groups cannot be obtained from a *set* of abelian groups building cartesian products, subgroups and extensions.

§1. Elementary propositions to approach the main problems of this paper

A. Some Necessary Definitions

Let C, D be arbitrary classes of right R-modules; then:

(a) $\mathbf{T}_C$ denotes the torsion class generated by C.

(b) ${}_C\mathbf{T}$ denotes the torsion class cogenerated by C.

(c) C_I denotes the smallest class of right R-modules which contains C and is closed under homomorphic images.

(d) ${}_KC$ denotes the smallest class of right R-modules which contains C and is closed under submodules.

(e) $C \leqq_{\mathbf{T}} D \Leftrightarrow \mathbf{T}_C \subset \mathbf{T}_D$,

$C \leqq_{\mathbf{F}} D \Leftrightarrow {}_C\mathbf{T} \subset {}_D\mathbf{T}$.

(f) We consider the following equivalence relations on C:

$M_R \sim_{\mathbf{T}} N_R \Leftrightarrow \mathbf{T}_{M_R} = \mathbf{T}_{N_R}$,

$M_R \sim_{\mathbf{F}} N_R \Leftrightarrow {}_{M_R}\mathbf{T} = {}_{N_R}\mathbf{T}$.

(g) Let $[M_R]_{\mathbf{T}}$ be an equivalence class of C with respect to "$\sim_{\mathbf{T}}$" and let $[N_R]_{\mathbf{F}}$ be an equivalence class of C with respect to "$\sim_{\mathbf{F}}$". We put:

$C\,|_{\sim_{\mathbf{T}}} := \{[M_R]_{\mathbf{T}} \mid M_R \in C\}$,

$C\,|_{\sim_{\mathbf{F}}} := \{[N_R]_{\mathbf{F}} \mid N_R \in C\}$.

B. Global Properties

LEMMA 1.1. (i) *If* Mod-$R \restriction_{\sim_T}$ *has small cardinality then every torsion class for* Mod-R *is generated by a single right* R*-module.*

(ii) *If* Mod-$R \restriction_{\sim_F}$ *has small cardinality then every torsion class for* Mod-R *is cogenerated by a single right* R*-module.*

PROOF. It is sufficient to prove property (i). Property (ii) follows by duality.

Let $\mathbf{T}'$ be an arbitrary torsion class for Mod-R; then $|\mathbf{T}' \restriction \sim_{\mathbf{T}}| \leqq |\text{Mod-}R \restriction \sim_{\mathbf{T}}|$. Hence $\mathbf{T}' \restriction \sim_{\mathbf{T}}$ has small cardinality. This implies that $\mathbf{T}'$ is generated by the direct sum of all right R-modules of $\mathbf{T}'$ which are not equivalent with respect to "$\sim_{\mathbf{T}}$".

PROPOSITION 1.1. *The following conditions are equivalent*:

(i) Mod-$R \restriction \sim_{\mathbf{T}}$ *has small cardinality.*

(ii) Mod-$R \restriction \sim_{\mathbf{F}}$ *has small cardinality.*

(iii) $\mathcal{T}$ *has small cardinality.*

(iv) *There exists a cardinal number* λ *such that for all non-trivial modules* M_R *the endomorphism ring* $\text{End}_R(M_R)$ *contains some non-trivial endomorphism* f *such that* $|\text{Im} f| \leqq \lambda$.

PROOF. It suffices to prove the equivalence of the conditions (i), (iii) and (iv).

(iii) $\Rightarrow$ (i): There exists a canonical injection $\varphi : \text{Mod-}R \restriction \sim_{\mathbf{T}} \rightarrow \mathcal{T}$ defined by

$$\varphi([M_R]_{\mathbf{T}}) = \mathbf{T}_{M_R} \qquad \text{for all } [M_R]_{\mathbf{T}} \in \text{Mod-}R \restriction \sim_{\mathbf{T}}.$$

(i) $\Rightarrow$ (iii): Because of Lemma 1.1 (i) every torsion class for Mod-R is generated by a single right R-module. Hence φ is a bijection.

(i) $\Rightarrow$ (iv): Let C be the set of all right R-modules which are not equivalent with respect to "$\sim_{\mathbf{T}}$". We put $\lambda := |\bigoplus_{N_R \in C} N_R|$.

We now consider an arbitrary non-trivial module M_R. There exists some $N_R \in C$ such that $N_R \sim_{\mathbf{T}} M_R \Rightarrow \text{Hom}_R(N_R, M_R) \neq \{0\}$.

Let f be a non-trivial element of $\text{Hom}_R(N_R, M_R)$; then $\text{Im} f \leqq_{\mathbf{T}} N_R \sim_{\mathbf{T}} M_R \Rightarrow \text{Hom}_R(M_R, \text{Im} f) \neq \{0\}$. Because of $|\text{Im} f| \leqq \lambda$ condition (iv) is proved.

(iv) $\Rightarrow$ (iii): Let $\mathbf{T}$ be an arbitrary torsion class for Mod-R. Let C be the set of all non-isomorphic modules $M_R \in \mathbf{T}$ such that $|M_R| \leqq \lambda$. Then $\mathbf{T}_C \subset \mathbf{T}$. C is closed under homomorphic images. Because of proposition 2.5 ([16], chapter vi) $\mathbf{T}_C$ consists of all modules N_R such that every non-trivial factor module of N_R has a non-trivial submodule in C. Hence condition (iv) implies that $\mathbf{T} \subset \mathbf{T}_C$ and hence $\mathbf{T} = \mathbf{T}_C$. If we now consider the set M of all non-isomorphic modules M_R such that $|M_R| \leqq \lambda$ then $|\mathcal{T}| \leqq 2^{|M|}$.

PROPOSITION 1.2. (i) *The following conditions are equivalent*:

(a) *All torsion classes for* Mod-R *are generated by single right* R*-modules.*

(b) *Every class* $\boldsymbol{C}$ *of right* R*-modules contains a (small) subset* $\boldsymbol{D}$ *of right* R*-modules such that* $\boldsymbol{C} \leqq_{\mathbf{T}} \boldsymbol{D}$.

(c) *Every class I contains a (small) subset K such that* $\bigvee_{i \in I} \mathbf{T}_i = \bigvee_{k \in K} \mathbf{T}_k$ *for arbitrary torsion classes* $\mathbf{T}_i$ $(i \in I)$ *for* Mod-R.

(ii) *The following conditions are equivalent*:

(d) *All torsion classes for* Mod-R *are cogenerated by single right* R*-modules.*

(e) *Every class* $\boldsymbol{C}$ *of right* R*-modules contains a (small) subset* $\boldsymbol{D}$ *of right* R*-modules such that* $\boldsymbol{D} \leqq_{\mathbf{F}} \boldsymbol{C}$.

(f) *Every class I contains a (small) subset K such that* $\bigwedge_{i \in I} \mathbf{T}_i = \bigwedge_{k \in K} \mathbf{T}_k$ *for arbitrary torsion classes* $\mathbf{T}_i$ $(i \in I)$ *for* Mod-R.

PROOF. Clearly it is sufficient to prove the equivalence of the conditions (a), (b) and (c).

(a) $\Rightarrow$ (b): Let $\boldsymbol{C}$ be an arbitrary class of right R-modules. $\mathbf{T}_C$ is generated by a single right R-module. Hence there exists some module M_R such that $\mathbf{T}_C = \mathbf{T}_{M_R} \Rightarrow \boldsymbol{C} \leqq_{\mathbf{T}} M_R$.

On the other hand, there exists for every submodule $N_R \subsetneqq M_R$ *one* module $N'_R \in \boldsymbol{C}$ such that $\mathrm{Hom}_R(N'_R, M_R/N_R) \neq \{0\}$. Let $\boldsymbol{D}$ be the class of all these modules; then $\boldsymbol{D}$ is a (small) set. Moreover, we may conclude that $M_R \leqq_{\mathbf{T}} \boldsymbol{D}$ (cf. [8], lemma 2(i)) $\Rightarrow \boldsymbol{C} \leqq_{\mathbf{T}} M_R \leqq_{\mathbf{T}} \boldsymbol{D}$.

(b) $\Rightarrow$ (c): Because of (b), $\bigvee_{i \in I} \mathbf{T}_i$ contains a (small) subset $\boldsymbol{D}$ such that $\bigvee_{i \in I} \mathbf{T}_i \leqq_{\mathbf{T}} \boldsymbol{D}$. Condition (c) now follows immediately.

(c) $\Rightarrow$ (a): Let $\mathbf{T}$ be an arbitrary torsion class for Mod-R $\Rightarrow T = \bigvee_{M_R \in \mathbf{T}} \mathbf{T}_{M_R}$. Because of (c) there exists a (small) subset $\boldsymbol{D} \subset \mathbf{T}$ such that $\mathbf{T} = \bigvee_{N_R \in \boldsymbol{D}} \mathbf{T}_{N_R} \Rightarrow T$ is generated by $\bigoplus_{N_R \in \boldsymbol{D}} N_R$.

C. Local Properties

Let $\boldsymbol{C}$ be an arbitrary class of right R-modules. We shall prove:

PROPOSITION 1.3. (i) *The following conditions are equivalent*:

(a) $\mathbf{T}_C$ *is generated by a single right* R*-module.*

(b) *There exists some cardinal number* λ *such that all non-trivial modules* $C_R \in \boldsymbol{C}_I$ *contain non-trivial submodules* $L_R \in \boldsymbol{C}_I$ *such that* $|L_R| \leqq \lambda$.

(ii) *The following conditions are equivalent*:

(c) ${}_C\mathbf{T}$ *is cogenerated by a single right* R*-module.*

(d) *There exists some cardinal number* λ *such that all non-trivial modules* $C_R \in {}_K\boldsymbol{C}$ *contain proper submodules* L_R *such that* $C_R/L_R \in {}_K\boldsymbol{C}$ *and* $|C_R/L_R| \leqq \lambda$.

PROOF. We only have to show the equivalence of the conditions (a) and (b).

(a) $\Rightarrow$ (b): Because of $\mathbf{T}_C = \mathbf{T}_{C_l}$ there exists some module $M_R \in \mathbf{T}_C$ such that $C_l \leqq_{\mathbf{T}} M_R$. On the other hand, there exists a (small) set $D \subset C_l$ such that $M_R \leqq_{\mathbf{T}} D$ (cf. the proof of Proposition 1.2). We put $\lambda := |\bigoplus_{N_R \subset D} N_R|$. The inequalities $C_l \leqq_{\mathbf{T}} M_R \leqq_{\mathbf{T}} D$ imply that $\mathrm{Hom}_R(\bigoplus_{N_R \in D} N_R, C_R) \cong \Pi_{N_R \in D} \mathrm{Hom}_R(N_R, C_R) \neq \{0\}$ for all modules $C_R \in C_l$. Condition (a) now follows immediately.

(b) $\Rightarrow$ (a): Let S_R be the direct sum of all non-isomorphic modules $M_R \in C_l$ such that $|M_R| \leqq \lambda$. The reader verifies immediately that it is sufficient to prove that $N_R \leqq_{\mathbf{T}} S_R$ for all modules $N_R \in C$. Because of ([8], lemma 2(i)) we thus have to show that $\mathrm{Hom}_R(S_R, N_R/L_R) \neq \{0\}$ for all non-trivial modules $N_R \in C$ and all submodules $L_R \subsetneqq N_R$. But for all non-trivial modules $N_R \in C$ and all submodules $L_R \subsetneqq N_R$ there exists some non-trivial module $D_R \in C_l$ such that $|D_R| \leqq \lambda$ and $D_R \subset N_R/L_R$. Hence (a) is shown.

PROPOSITION 1.4. (i) *The following conditions are equivalent:*

(a) $C|_{\sim_{\mathbf{T}}}$ *has small cardinality.*

(b) *There exists a cardinal number λ such that for every non-trivial module $M_R \in C$ and every submodule $L_R \subsetneqq M_R$ there exists some non-trivial homomorphism $f: M_R \to M_R/L_R$ such that $|\mathrm{Im} f| \leqq \lambda$.*

(ii) *The following conditions are equivalent:*

(c) $C|_{\sim_{\mathbf{F}}}$ *has small cardinality.*

(d) *There exists a cardinal number λ such that for every non-trivial module $M_R \in C$ and every non-trivial submodule $L_R \subset M_R$ there exists some non-trivial homomorphism $f: L_R \to M_R$ such that $|\mathrm{Im} f| \leqq \lambda$.*

PROOF. We only prove the equivalence of the conditions (a) and (b).

(a) $\Rightarrow$ (b): Let D be the set of all right R-modules $C_R \in C$ which are not equivalent with respect to "$\sim_{\mathbf{T}}$". We put $\lambda := |\bigoplus_{N_R \in D} N_R|$.

In the next step we consider an arbitrary non-trivial module $M_R \in C$ and some submodule $L_R \subsetneqq M_R$. Because of $M_R/L_R \leqq_{\mathbf{T}} M_R$ there exists some module $N_R \in D$ such that $M_R \sim_{\mathbf{T}} N_R$ and $\mathrm{Hom}_R(N_R, M_R/L_R) \neq \{0\}$. The desired conclusion now follows immediately.

(b) $\Rightarrow$ (a): Let M_R be a non-trivial module of C. Because of condition (b) there exists for every submodule $L_R \subsetneqq M_R$ some non-trivial homomorphism $f_{L_R}: M_R \to M_R/L_R$ such that $|\mathrm{Im} f_{L_R}| \leqq \lambda$. We put $N_R := \bigoplus \mathrm{Im} f_{L_R}$. It is easily seen that $N_R \leqq_{\mathbf{T}} M_R$. On the other hand, lemma 2(i) of [8] implies that $M_R \leqq_{\mathbf{T}} N_R \Rightarrow M_R \sim_{\mathbf{T}} N_R$.

Let M be the set of all non-isomorphic modules H_R such that $|H_R| \leqq \lambda$. We may conclude that $|C|\sim_{\mathbf{T}}| \leqq 2^{|M|}$ (cf. the proof of Proposition 1.1).

Let $\boldsymbol{P}$ be the class of projective and let $\boldsymbol{E}$ be the class of injective right R-modules. We obtain the following

COROLLARY 1.1. *$\boldsymbol{P}\,|\!\sim_{\mathrm{T}}$, $\boldsymbol{P}\,|\!\sim_{\mathrm{F}}$, $\boldsymbol{E}\,|\!\sim_{\mathrm{T}}$ and $\boldsymbol{E}\,|\!\sim_{\mathrm{F}}$ have small cardinality.*

PROOF. In order to prove that $\boldsymbol{P}\,|\!\sim_{\mathrm{T}}$ and $\boldsymbol{P}\,|\!\sim_{\mathrm{F}}$ have small cardinality one may use Proposition 1.4 and a theorem of Kaplansky [11] which says that every projective right R-module is the direct sum of projective right R-modules which are generated by countable many elements. On the other hand, for every non-trivial injective right R-module E_R and every $x \in E_R$, $E(xR)$ is a direct summand of E_R. Hence Proposition 1.4 implies that $\boldsymbol{E}\,|\!\sim_{\mathrm{T}}$ and $\boldsymbol{E}\,|\!\sim_{\mathrm{F}}$ have small cardinality.

Corollary 1.1 implies in particular that every torsion class for Mod-R which is generated or cogenerated by a class of projective or injective right R-modules is generated or cogenerated by a single right R-module.

§2. A construction of almost free abelian groups to solve the main problems of this paper

For the convenience of the reader we will list some notations of set and abelian group theory.

An ordinal α is always identified with the set $\{\beta \mid \beta$ ordinal, $\beta < \alpha\}$. A cardinal κ is an initial ordinal, i.e. $\kappa = \inf\{\alpha \mid |\alpha| = \kappa\}$. If λ is a (limit) ordinal and $C \subseteq \lambda$ such that $\lambda = \sup C$ and C closed with respect to the wellordering of α then C is a *cub*. A subset $S \subset \lambda$ is *stationary* in λ if $S \cap C \neq \emptyset$ for all cubs $C \subset \lambda$. A stationary set $S \subset \lambda$ is *sparse* if $S \cap \mu$ is not stationary in μ for all limit ordinals $\mu < \lambda$. If A is a set of cardinality κ a family $\{A_\nu \mid \nu < \kappa\}$ of subsets of A is called *κ-filtration* of A if $A_\nu \subset A_\tau$ for all $\nu < \tau < \kappa$, $A_\lambda = \bigcup_{\alpha<\lambda} A_\alpha$ for limit ordinals λ and $|A_\alpha| < \kappa$ for all $\alpha < \kappa$ and $A = \bigcup_{\alpha<\kappa} A_\alpha$. We will use the weak diamond principle $\Phi_\kappa(S)$ introduced by Devlin and Shelah [1].

If κ is a cardinal and S a (stationary) subset of κ then $\Phi_\kappa(S)$ is the following statement:

$\Phi_\kappa(S)$: Let A be a set of cardinality κ and let $\{A_\nu \mid \nu < \kappa\}$ be a κ-filtration of A. If for $\nu \in S$ a partition $P_\nu : 2^{A_\nu} \to \{0,1\}$ of the subsets of A_ν into two classes is given, then there exists a map $\varphi : S \to \{0,1\}$ such that for all $X \subseteq A = \bigcup_{\nu<\kappa} A_\nu$ the set $\{\nu \in S \mid P_\nu(X \cap A_\nu) = \varphi(\nu)\}$ is stationary in κ.

If $S \subset \kappa$ and $\Phi_\kappa(S)$ holds, then S is called *non-small*. In the constructible universe L, $\Phi_\kappa(S)$ holds for all uncountable, regular κ and all stationary subsets of κ. Devlin and Shelah [1] proved that $2^{\aleph_0} < 2^{\aleph_1}$ implies $\Phi_{\aleph_1}(\aleph_1)$. If κ is a regular cardinal we consider the principle ∇_κ; cf. Dugas and Göbel [3] and Shelah [15, theorem 9].

∇_κ: There exists a subset $S \subset \kappa$ such that
(i) $S \subset \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \aleph_0\}$.
(ii) S is *non-small*; i.e. $\Phi_\kappa(S)$ holds.
(iii) S is sparse.

This principle was used by Shelah [15] and Dugas and Göbel [3] to construct strongly κ-free modules. If we assume the generalized continuum hypothesis GCH and $\kappa = \mu^+$, $\mathrm{cf}(\mu) \neq \aleph_0$ is a successor cardinal we can omit (ii), cf. Shelah [15, remark 10] where he also states that any non-small set $S \subset \kappa$ can be partitioned into κ non-small, disjoint sets.

It is known that ∇_κ is a consequence of GCH + (There exists no inner model of ZFC with measurable cardinals) for arbitrary large κ; cf. A. Dodd and R. Jensen [2].

Our terminology is as in Fuchs ([5], [6]). A non-trivial abelian group G is *κ-free*, κ a cardinal, if all subgroups $B \subseteq G$ with $|B| < \kappa$ are free. G is *strongly κ-free* if G is κ-free and each subgroup B of cardinality $< \kappa$ is contained in a subgroup C of cardinality $< \kappa$ such that G/C is κ-free. If p is a prime number we denote by $\mathbf{Z}(p)$ the cyclic group of order p, J_p is the ring of p-adic integers and $\mathbf{Q}_p$ is the set of all rationals whose denominator is relatively prime to p. An abelian group G is *cotorsion-free* if 0 is the only cotorsion subgroup of G. Hence $G \neq 0$ is cotorsion-free if no subgroup of G is isomorphic to $\mathbf{Q}$, $\mathbf{Z}(p)$ or J_p for any prime p (cf. Dugas and Göbel [3]) and a countable group is cotorsion-free if it is torsion free and reduced. It is well known that each cotorsion group $\neq 0$ contains a pure-injective one. Hence $\mathrm{Hom}(A, G) \neq \{0\}$ if G is *not* cotorsion-free and A contains $\mathbf{Z}$ as a pure subgroup. This implies $\mathrm{Hom}(A, G) \neq \{0\}$ for all (strongly) κ-free A and *not* cotorsion-free $G \neq 0$. In contrast to this we will show — using ∇_κ — that there exists a strongly κ-free abelian group A of cardinality κ such that $\mathrm{Hom}(A, G) = \{0\}$ for all cotorsion-free G of cardinality $< \kappa$.

We will use $A \sqsubset B$ to indicate that A is a direct summand of the abelian group B.

We need the following

Lemma 2.1 (Step-Lemma). *Let $F = \bigcup_{n<\omega} F_n$ be a free abelian group and $F_0 \subsetneqq F_1 \subsetneqq \cdots$ a chain of direct summands. We fix elements $\pi \in \hat{\mathbf{Z}}$, $e \in F_0$. Then there exist two free extensions $F^\varepsilon(e, \bigcup_{n<\omega} F_n, \pi) = F^\varepsilon \supset F$, $\varepsilon \in \{0,1\}$ such that*

(i) $F_n \sqsubset F^\varepsilon$ *for all* $n < \omega$ *and* $\varepsilon \in \{0,1\}$.

(ii) *If* G *is a reduced and torsion-free abelian group and* $\varphi \in \mathrm{Hom}(F,G)$, $\varphi^\varepsilon \in \mathrm{Hom}(F^\varepsilon, G)$, $\varepsilon \in \{0,1\}$, *such that* $\varphi^\varepsilon \restriction_F = \varphi$ *then* $\pi e^\varphi \in G$ *(computed in the* **Z**-*adic completion of the Hausdorff group* G*).*

PROOF. W.l.o.g. we can assume $e = e_0 \in B_0$ where B_0 is a basis of F_0. Let B_n be a basis of a complement of F_{n-1} in F_n, i.e. $F_n = F_{n-1} \oplus \langle B_n \rangle$. We choose $e_n \in B_n$ and set $D = \langle B_n - \{e_n\} \mid n < \omega \rangle$. Hence we have $F = D \oplus \langle e_n \mid n < \omega \rangle$. We fix a representation $\pi = \sum_{i=0}^{\infty} \pi_i i!$, $\pi_i \in \mathbf{Z}$. Let $F^\varepsilon = D \oplus \langle e_0 \rangle \oplus \langle f_i^\varepsilon \mid 1 \leq i < \omega \rangle$ with free generators f_i^ε. We define a homomorphism $\varepsilon^* : F \to F^\varepsilon$, $\varepsilon \in \{0,1\}$, by $\varepsilon^* \restriction_{D \oplus \langle e_0 \rangle} = \mathrm{id}$ and $e_n^{\varepsilon^*} = n f_{n+1}^\varepsilon - f_n^\varepsilon + \pi_{n-1}^\varepsilon e_0$ for all $1 \leq n < \omega$ where $\pi_i^0 = 0$ and $\pi_i = \pi_i^1$. One easily verifies that ε^* is an embedding satisfying (i). We will identify F and $\varepsilon^*(F)$, $\varepsilon \in \{0,1\}$.

Let G be an abelian group, $\varphi \in \mathrm{Hom}(F,G)$, $\varphi^\varepsilon \in \mathrm{Hom}(F^\varepsilon, G)$ with $\varphi = \varphi^\varepsilon \restriction_F$. Setting $z_n = \varphi^1(f_n^1) - \varphi^0(f_n^0) \in G$ and applying φ we obtain from the definition of ε^* the equations:

$$n z_{n+1} - z_n = (-\pi_{n-1}^1 - \pi_n^0) e_0^\varphi = -\pi_{n-1} e_0^\varphi.$$

By induction we get

$$n!\, z_{n+1} = z_1 - \left(\sum_{i=0}^{n-1} \pi_i i! \right) e_0^\varphi.$$

Hence $z_1 = e_0 \pi \in G$ computed in the **Z**-adic closure $\hat{G}$ of G.

Our "Step-Lemma" enables us to prove the

THEOREM 2.1. *Let* κ *be a regular cardinal such that* ∇_κ *holds. Then there exist* 2^κ *strongly* κ-*free abelian groups* A *of cardinality* κ *such that* $\mathrm{Hom}(A,G) = \{0\}$ *for all cotorsion-free abelian groups* G *with* $|G| < \kappa$.

PROOF. Set $\tilde{\kappa} = \min\{\kappa, 2^{\aleph_0}\}$. If $\tilde{\kappa} = 2^{\aleph_0}$ we set $I = \hat{\mathbf{Z}}$. If $\kappa = \tilde{\kappa}$ we choose a pure subgroup I_p of size κ of J_p containing $\mathbf{Z}$ and we set $I = \bigcup_p I_p$ where p runs over all prime numbers. Let $I = \{\pi^\alpha \mid \alpha < \tilde{\kappa}\}$ be an enumeration of I. Let A be a set of cardinality κ and $A = \bigcup_{\nu<\kappa} A_\nu$ a κ-filtration of A. Let S be the sparse, non-small subset of κ given by ∇_κ and remember that $\mathrm{cf}(\lambda) = \omega$ for all $\lambda \in S$. We can partition S in non-small disjoint subsets $S_{\beta\alpha}$ for $\beta < \kappa$, $\alpha < \tilde{\kappa}$. Choose $A = \{a_\beta \mid \beta < \kappa\}$ an enumeration of A. We can assume that $a_\beta \in A_\nu$ if $\nu \in S_{\beta\alpha}$. If $\nu \in S_{\beta\alpha}$ we will choose successor ordinals $\nu_n < \nu_{n+1}$ such that $\nu = \sup\{\nu_n \mid n < \omega\}$ and $a_\beta \in A_{\nu_0}$. Let $O_\nu = A_\nu \times A_\nu \times A_\nu$ and $U_\nu = O_\nu \cup A_\nu$ a disjoint union. For all $\nu \in S_{\beta\alpha}$ we will define a partition $P_{\beta\alpha} : 2^{U_\nu} \to \{0,1\}$ of the power set of U_ν:

If $X \subseteq U_\nu$, $\nu \in S_{\beta\alpha}$ we set $P_{\beta\alpha}(X) = 0$ iff

(1) $X \cap O_\nu$ defines a binary operation $+_X$ on A_ν such that $A_\nu(+_X)$ is a free abelian group and $A_{\nu_n} \sqsubset A_\nu$ for all n.

(2) $X \cap A_\nu$ is a subgroup of $A(+_X)$.

(3) If $\psi : A_\nu \to A_\nu/(X \cap A_\nu)$ is the natural projection, then ψ does not lift to a homomorphism

$$\psi' : A_\nu^0\left(a_\beta, \bigcup_{n<\omega} A_{\nu_n}, \pi^\alpha\right) \to A_\nu/(X \cap A_\nu).$$

Let $\varphi_{\beta\alpha} : S_{\beta\alpha} \to \{0,1\}$ be the function given by $\Phi_\kappa(S_{\beta\alpha})$. We define an abelian group structure on A such that:

(1) A_ν is a free subgroup of A for all $\nu < \kappa$.

(2) If $\lambda < \kappa$ is a limit ordinal, $A_\lambda = \bigcup_{\alpha<\lambda} A_\alpha$ as a group.

(3) If $\alpha \in \kappa \setminus S$ then $A_\alpha \sqsubset A_\tau$ for all $\alpha < \tau < \kappa$.

(4) If $\nu \in S_{\beta\alpha}$ then $A_{\nu+1} = A_\nu^{\varphi_{\beta}(\nu)}(a_\beta, \bigcup_{n<\omega} A_{\nu_n}, \pi^\alpha)$.

This can be done (cf. Eklof and Mekler [4] or Dugas and Göbel [3]) and A is strongly κ-free.

Let G be an abelian, cotorsion-free group of cardinality $<\kappa$ and $\varphi \in \operatorname{Hom}(A, G)$. Set $C = \{\nu < \kappa \mid \varphi(A_\nu) = \varphi(A)\}$. Since κ is regular and $|G| < \kappa$ the set C is a cub in κ. Let $K = \operatorname{Ker} \varphi$. Since $S_{\beta\alpha}$ is non-small, all sets

$$\tilde{S}_{\beta\alpha} = \{\nu \in S_{\beta\alpha} \mid P_{\beta\alpha}((+_A \cup K) \cap U_\nu) = \varphi_{\beta\alpha}(\nu)\}$$

are stationary in κ. Choose $\lambda \in C \cap \tilde{S}_{\beta\alpha}$. We get a diagram

$$\begin{array}{ccccl} A_{\lambda+1} & \overset{\varphi \mid A_{\lambda+1}}{\searrow} & & & \\ \cup & & & & \\ A_\lambda & \xrightarrow[\varphi \mid A_\lambda]{} & G & \xrightarrow{\tau^{-1}} & A_\lambda \mid K_\lambda, \quad K_\lambda = K \cap A_\lambda \end{array}$$

and $\tau : A_\lambda \mid K_\lambda \to A^\varphi$ is the natural isomorphism defined by $(a + K_\lambda)^\tau = a^\varphi$. The composition map $\psi = \varphi \mid A_\lambda^{\tau^{-1}} : A_\lambda \to A_\lambda \mid \kappa_\lambda$ is the natural projection which by the diagram lifts to $A_{\lambda+1} = A_\lambda^{\varphi_{\beta\alpha}(\lambda)}(a_\beta, \bigcup_{n<\omega} A_{\lambda_n}, \pi^\alpha)$. Hence $\varphi_{\beta\alpha}(\lambda) = 1$ and ψ lifts to both extensions of A_λ. Hence the "Step-Lemma" implies that $a_\beta^\varphi \pi^\alpha \in G$ for all $\beta < \kappa$, $\alpha < \tilde{\kappa}$. Let $\widehat{G/p^\omega G}$ be the p-adic completion of $G/p^\omega G$.

Case I: $\kappa < 2^{\aleph_0}$. Here we have $\tilde{\kappa} = \kappa$ and $|G| < \kappa$ implies that the map $\sigma : I_p \to G/p^\omega G$ with $x^\sigma = (a_\beta^\varphi + p^\omega G)x$ is not one to one. But $G/p^\omega G$ is "Hausdorff" in the p-adic topology and therefore σ lifts to a J_p-homomorphism $\sigma' : J_p \to \widehat{G/p^\omega G}$ whose kernel is an ideal $\neq 0$ of J_p. Hence $J_p^{\sigma'}$ is a torsion subgroup in the torsion-free group $\widehat{G/p^\omega G}$. So $\sigma' = 0$ which implies that $a_\beta^\varphi \in p^\omega G$ for all $\beta < \kappa$ and all primes p. Hence $\varphi = 0$.

Case II: $\kappa \geqq 2^{\aleph_0}$. In this case we have $a_\beta^\varphi \pi^\alpha \in G$ for all $\beta < \kappa$, $\alpha < 2^{\aleph_0}$ and $I = \hat{\mathbf{Z}}$. Hence $\pi^\alpha \mapsto a_\beta^\varphi \pi^\alpha$ is a homomorphism from $\hat{\mathbf{Z}}$ into G. But G is cotorsion-free and hence $\mathrm{Hom}(\hat{\mathbf{Z}}, G) = 0$. Therefore we get $a_\beta^\varphi = 0$ for all $\beta < \kappa$. So we have finished the proof that A has the desired property. To obtain 2^κ such groups the corresponding argument in Eklof and Mekler [4] can be used.

We now assume that ∇_κ *holds for arbitrary large (regular) cardinals* κ. We denote this axiom by ∇.

The following corollary solves the main problems of this paper:

COROLLARY 2.1. (∇)

(i) *The class* $\mathcal{T}_G$ *of torsion classes for* Mod-**Z** *which are not generated by single abelian groups and the class* $\mathcal{T}_C$ *of torsion classes for* Mod-**Z** *which are not cogenerated by single abelian groups have no small cardinality.*

(i′) *For the class* **F** *of cotorsion-free abelian groups we may conclude especially that* ${}_{\mathbf{F}}\mathbf{T}$ *is not cogenerated by a single abelian group (this answers the conjecture of Göbel and Wald* [7] *positively).*

(ii) *The class* $\mathcal{T}_S$ *of torsion classes for* Mod-**Z** *which are generated by single abelian groups and the class* $\mathcal{T}_p$ *of torsion classes for* Mod-**Z** *which are cogenerated by single abelian groups have no small cardinality.*

PROOF. Because of Proposition 1.1 (ii) is a consequence of (i). Hence we only have to prove (i) and (i′).

1. $\mathcal{T}_G$ *has no small cardinality*

Because of our set theoretical assumption and Theorem 2.1 there exist for arbitrary large regular cardinals κ strongly κ-free abelian groups A_κ of cardinality κ such that $\mathrm{Hom}(A_\kappa, G) = \{0\}$ for all cotorsion-free abelian groups G with $|G| < \kappa$. Thus we may conclude especially that $\mathrm{Hom}(A_\kappa, A_{\tilde\kappa}) = \{0\}$ for regular cardinals $\tilde\kappa < \kappa$ such that $\nabla_{\tilde\kappa}$ and ∇_κ hold.

Let $\mathbf{G}_\kappa$ be the class of cotorsion-free abelian groups of cardinality $< \kappa$, then ${}_{G_\kappa}\mathbf{T} \subsetneqq {}_{G_{\tilde\kappa}}\mathbf{T}$ for regular cardinals $\tilde\kappa < \kappa$ such that $\nabla_{\tilde\kappa}$ and ∇_κ hold. Hence $\mathcal{T}_F := \{{}_{G_\kappa}\mathbf{T} \mid \kappa$ a regular cardinal such that ∇_κ holds$\}$ has no small cardinality and it is sufficient to show that ${}_{G_\kappa}\mathbf{T}$ is not generated by a single abelian group for all ${}_{G_\kappa}\mathbf{T} \in \mathcal{T}_F$. If ${}_{G_\kappa}\mathbf{T} = \mathbf{T}_M$ for some abelian group M then we consider for a (regular) cardinal $\tilde\kappa > \max\{|M|, \kappa\}$ a (strongly) κ-free abelian group $A_{\tilde\kappa}$ such that $\mathrm{Hom}(A_{\tilde\kappa}, G) = \{0\}$ for all $G \in \mathbf{G}_\kappa \Rightarrow A_{\tilde\kappa} \in \mathbf{T}_M \Rightarrow \mathrm{Hom}(M, A_{\tilde\kappa}) \neq \{0\} \Rightarrow M = \mathbf{Z} \oplus M'$ for some abelian group $M' \Rightarrow \mathbf{T}_M =$ Mod-**Z**, which is impossible.

We may now easily verify (i′).

If ${}_{\mathbf{F}}\mathbf{T}$ is cogenerated by a single abelian group then ${}_{\mathbf{F}}\mathbf{T} = {}_{G_{\tilde\kappa}}\mathbf{T}$ for some

${}_{G_{\tilde{\kappa}}}\mathbf{T} \in \mathcal{T}_F \Rightarrow {}_{G_\kappa}\mathbf{T} = {}_{G_{\tilde{\kappa}}}\mathbf{T}$ for all $\tilde{\kappa} < \kappa$, which is a contradiction. The pair $({}_{\mathbf{F}}\mathbf{T}, \mathbf{F})$ is a torsion-theory for Mod-**Z** (cf. [16], chapter vi). Hence (i′) implies in particular that there exists no (small) set $\boldsymbol{D} \subset \mathbf{F}$ such that $\mathbf{F}$ is the smallest class of abelian groups containing $\boldsymbol{D}$ which is closed under extensions, subgroups and direct products. This answers the conjecture of Göbel and Wald [7] positively.

2. *$\mathcal{T}_C$ has no small cardinality*

Let κ be a regular cardinal such that ∇_κ holds. We consider the class $\boldsymbol{S}_\kappa := \{A \in \text{Mod-}\mathbf{Z} \mid |A| \geqq \kappa,\ A$ is κ-free and $\text{Hom}(A, G) = \{0\}$ for all cotorsion-free abelian groups G with $|G| < \kappa\}$. With the help of Theorem 2.1 the reader will immediately verify that ${}_{S_{\tilde{\kappa}}}\mathbf{T} \subsetneqq {}_{S_\kappa}\mathbf{T}$ for regular cardinals $\tilde{\kappa} < \kappa$ such that $\nabla_{\tilde{\kappa}}$ and ∇_κ hold. Hence the class $\mathcal{T}_L$ of all torsion classes ${}_{S_\kappa}\mathbf{T}$ for Mod-**Z** such that ∇_κ holds has no small cardinality and it is sufficient to show that ${}_{S_\kappa}\mathbf{T}$ is not cogenerated by a single abelian group for all (regular) cardinals κ such that ∇_κ holds.

If ${}_{S_\kappa}\mathbf{T}$ is cogenerated by a single abelian group then there exists a (small) set $\boldsymbol{U} \subset \boldsymbol{S}$ such that ${}_{S_\kappa}\mathbf{T} = {}_{U}\mathbf{T}$ (cf. Proposition 1.2, the implication (d) $\Rightarrow$ (e)). On the other hand, Theorem 2.1 implies the existence of some $A \in \boldsymbol{S}_\kappa$ such that $\text{Hom}(A, U) = \{0\}$ for all $U \in \boldsymbol{U}$ which contradicts the fact that $\boldsymbol{U} \leqq_{\mathbf{F}} A$. Hence ${}_{S_\kappa}\mathbf{T}$ is not cogenerated by a single abelian group.

We should mention that Eklof and Mekler [4] proved that there exist indecomposable strongly κ-free abelian groups of cardinality κ for all regular not weakly compact cardinals κ if we assume all sets to be constructible. This set-theoretical assumption allows one to prove that ∇_κ holds for arbitrary large (regular) cardinals κ (cf. [11] for details). Eklof and Mekler's theorem allows one, in particular, to show that ${}_{\mathbf{Z}}\mathbf{T}$ is not generated by a single abelian group (cf. Corollary 2.1(i)).

On the other hand, Eklof and Mekler's theorem does not allow one to prove either that $\mathcal{T}_G$ has no small cardinality, or the existence of torsion classes for Mod-**Z** which are not cogenerated by single abelian groups. Hence we looked for a theorem which enabled us to prove these results as well.

§3. Some supplementary remarks

(1) We assume the same set theory as in §2 (i.e. ∇_κ holds for arbitrary large (regular) cardinals κ). Let $(\mathcal{T}, \subset)$ be the lattice of arbitrary torsion classes for Mod-**Z**. Remark 3.3.1 of [8] implies that $(\mathcal{T}, \subset)$ contains a uniquely determined coatom $\mathbf{T}_c$ which is cogenerated by **Z**. ${}_{\mathbf{Z}}\mathbf{T}$ is not generated by a single abelian group (cf. the proof of Corollary 2.1(i)). Because of proposition 6 of [8] the atoms

of $(\mathcal{T}, \subset)$ are generated by the irreducible divisible torsion groups. Let $\boldsymbol{O}$ be the set of all non-isomorphic simple abelian groups and let $\boldsymbol{P}$ be the set of all non-isomorphic irreducible divisible torsion groups. We now consider an arbitrary atom $\mathbf{T}_D$ $(D \in \boldsymbol{P})$ of $(\mathcal{T}, \subset)$. Because of the well known fact that an abelian group is divisible if it contains no maximal subgroups, it is easy to see that $\mathbf{T}_D$ is cogenerated by $(\Pi_{A \in \boldsymbol{O}} A) \oplus (\Pi_{E \in \boldsymbol{P} \setminus \{D\}})$. Hence the atoms of $(\mathcal{T}, \subset)$ are generated and cogenerated by single abelian groups.

(2) Let C be an arbitrary *connected* category. A (general) torsion theory $(\mathbf{T}, \mathbf{F})$ for C may be defined by replacing the "orthogonality axiom" for abelian categories by the axiom: $\mathrm{Hom}_C(T, F)$ *consists only of constant morphisms* for all $T \in \mathbf{T}$, $F \in \mathbf{F}$ (cf. Preuß [14], Bemerkungen 5.1.26).

We now consider the category Top of non-empty topological spaces. The class $\mathbf{F}$ of T_1-spaces is a torsion free class of topological spaces in the above sense (cf. [14], Satz 6.2.4). The torsion class $\mathbf{T}$ of topological spaces cogenerated by $\mathbf{F}$ is not cogenerated by a single T_1-space. This is an immediate consequence of the following theorem of Herrlich [9]:

The following conditions are equivalent for a topological space Y:

(i) Y *is a* T_1*-space.*

(ii) *There exists a regular space* X *of at least two elements such that every continuous map* $f: X \to Y$ *is constant.*

In particular, we may conclude that the class $\mathcal{T}_{\mathrm{Top}}$ of (arbitrary) torsion classes for Top has *no small* cardinality (cf. the proof of Lemma 1.1).

(3) We now consider connected categories C which satisfy the following conditions:

T1: Every torsion class for C which is generated or cogenerated by a (small) set of objects is generated or cogenerated by a single object.

T2: If $M \leqq_{\mathbf{T}} \boldsymbol{S}$, $\boldsymbol{S}' \leqq_{\mathbf{F}} N$ for objects $M, N \in C$ and subclasses $\boldsymbol{S}, \boldsymbol{S}' \subset C$ then there exist (small) subsets $\boldsymbol{U} \subset \boldsymbol{S}$, $\boldsymbol{U}' \subset \boldsymbol{S}'$ such that $M \leqq_{\mathbf{T}} \boldsymbol{U}$ and $\boldsymbol{U}' \leqq_{\mathbf{F}} N$.

Clearly Mod-R and Top satisfy the conditions T1 and T2 (cf. the proof of Proposition 1.2 ((a) $\Rightarrow$ (b))).

In order to formulate the next proposition we consider connected categories C and C' which satisfy the conditions T1 and T2. Let $(\mathcal{T}_C, \subset)$ and $(\mathcal{T}_{C'}, \subset)$ be the lattices of torsion classes for C and C'. We obtain the following remarkable

Proposition 3.1. *If* $\Phi: (\mathcal{T}_C, \subset) \to (\mathcal{T}_{C'}, \subset)$ *is an isomorphism of complete* *lattices then a torsion class* $\mathbf{T} \in \mathcal{T}_C$ *is generated or cogenerated by a single object iff* $\Phi(\mathbf{T})$ *is generated or cogenerated by a single object.*

PROOF. It is sufficient to prove that $\Phi(\mathbf{T})$ is generated by a single object if $\mathbf{T}$ is generated by a single object. Let $\mathbf{T}$ be generated by $M \in C$. We put $\Phi(\mathbf{T}) := \mathbf{T}'$. Because of $\mathbf{T}' = \bigvee_{K \in \mathbf{T}} \mathbf{T}_K$ we may conclude that

$$\mathbf{T} = \mathbf{T}_M = \bigvee_{K \in \mathbf{T}'} \Phi^{-1}(\mathbf{T}_K) \Rightarrow M \leqq_{\mathbf{T}} \cup \{\Phi^{-1}(\mathbf{T}_K) \mid K \in \mathbf{T}'\} =: S.$$

T2 implies the existence of a (small) subset $U \subset S$ such that $M \leqq_{\mathbf{T}} U$. Hence there exists a (small) subset $U' \subset \mathbf{T}'$ such that

$$\mathbf{T} = \mathbf{T}_M = \bigvee_{K \in U'} \Phi^{-1}(\mathbf{T}_K) \Rightarrow \Phi(\mathbf{T}) = \mathbf{T}' = \bigvee_{K \in U'} \mathbf{T} = \mathbf{T}_U.$$

Because of T1 there exists an object $M' \in C'$ such that $\mathbf{T}' = \mathbf{T}_{M'}$.

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